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The D-branes of $\mathrm{SU}(n)$

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Abstract: D-branes that appear to generate all the K-theory charges of string theory on $\mathrm{SU}(n)$ are constructed, and their charges are determined.

Keywords: D-branes, Conformal Field Models in String Theory, Conformal and W Symmetry.

Contents

1. Introduction

It is widely believed that the charges of D-branes can be described in terms of (twisted) K-theory [1, 2, 3]. For example, for the case of string theory on the simply connected group G, the charge group is conjectured to be the twisted K-group of G (for more details see for example [4, 5]). Modulo some technical assumption, this twisted K-group has been calculated in [5] (see also [6, 7]) to be

$$ {}^{k+h^\vee}K^*(G) \cong \mathbb{Z}_{d(G,k)}^{\oplus m}\,, \qquad m = 2^{\mathrm{rk}(\bar{\mathfrak{g}})-1}\,, \tag{1.1}$$

where $h^\vee$ is the dual Coxeter number of G, k is the level of the underlying WZW model, and $d(G,k)$ is the integer

$$ d(G,k) = \frac{k+h^\vee}{\gcd(k+h^\vee, L)}\,, \tag{1.2}$$

where L only depends on $\bar{\mathfrak{g}}$, the finite dimensional Lie algebra associated to $\mathfrak{g}_k$. The summands $\mathbb{Z}_{d(G,k)}$ are equally divided between even and odd degree if $\mathrm{rk}(\bar{\mathfrak{g}}) > 1$. For $\bar{\mathfrak{g}} = \mathfrak{su}(2)$ the only summand is in even degree.

On the other hand, D-branes can be constructed in terms of the underlying conformal field theory, and it should be possible to determine their charges using this microscopic description. In particular, it was shown in [8] how the charges for branes that preserve the affine algebra $\mathfrak{g}_k$ (up to an automorphism) can in principle be calculated. For the D-branes that preserve the full algebra without any automorphism, the charges were then determined for SU(N) [8, 9], and later for all simply-connected Lie groups [10], and it was found that they account precisely for one summand $\mathbb{Z}_{d(G,k)}$. More recently, the charges of the D-branes that preserve the affine algebra up to an outer automorphism were determined [11]; whenever these twisted D-branes exist, they also contribute one summand $\mathbb{Z}_{d(G,k)}$. These

D-branes therefore only account at most for two of the summands $\mathbb{Z}_{d(G,k)}$ in (1.1); a CFT description of D-branes carrying charges which lie in the remaining summands is still missing.

In [9] it was suggested that at least some of the remaining D-branes could be described by branes that only preserve the affine algebra $\mathfrak{h}$ associated to a Cartan subalgebra $\bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$, together with the symmetry of the coset $\mathfrak{g}_k/\mathfrak{h}$. The affine algebra $\mathfrak{h}$ is obviously just the algebra of $r = \mathrm{rk}(\bar{\mathfrak{g}})$ free bosons, and one can choose r Neumann or Dirichlet boundary conditions independently from one another. There are therefore something like 2^r different gluing conditions that can be generated in this manner, and one may expect that D-branes with these gluing conditions can account for the full charge group (1.1). However, although such D-branes were already discussed in [12], the details of this proposal were not worked out. Furthermore, it was not clear how to calculate the charges that are produced by these branes.

In this paper we shall give a detailed construction of these branes, extending techniques given in [13, 14]. Furthermore we shall explain how to determine the associated charges, and we shall find that they account precisely for the missing summands in (1.1). We shall only consider the case of A_n in this paper, but we expect that our construction will generalise uniformly to the other cases as well.

Most of our discussion will be phrased in terms of the purely bosonic theory, but it is easy to see that the branes we construct actually preserve the $N = 1$ worldsheet supersymmetry (provided we choose the corresponding boundary conditions for the fermions); they therefore give rise to spacetime supersymmetric branes.

The paper is organised as follows. In the next section we review the WZW models, and in particular explain how the space of states can be written in terms of representations of the coset algebra $\mathfrak{g}_k/\mathfrak{h}$ and the algebra $\mathfrak{h}$. The boundary states are constructed in section 3, and in section 4 we argue that these branes actually generate the full K-theory group.

2. Some details about WZW-models

String theory on a group manifold G is described in terms of representations of the affine Lie algebra $\mathfrak{g}$ at level k. For the situation where the group manifold is simply-connected (a systematic analysis of the D-brane charges on non simply-connected manifolds was recently started in [15]), the full spectrum of the theory is then

$$\mathcal{H} = \bigoplus_{\lambda \in P_+^k} \mathcal{H}_\lambda \otimes \bar{\mathcal{H}}_{\lambda^*} \,, \tag{2.1}$$

where the sum runs over all integrable highest weight representations λ of $\mathfrak{g}_k$, and the representation for the right-movers is conjugate to the representation of the left-movers. (This theory is therefore sometimes referred to as the 'charge-conjugation' theory.)

We are interested in constructing boundary conditions that only preserve the subalgebra $\mathfrak{h}$ that is associated to a Cartan subalgebra $\bar{\mathfrak{h}} \subset \bar{\mathfrak{g}}$, as well as the corresponding coset algebra $\mathfrak{g}_k/\mathfrak{h}$. For this purpose it is convenient to decompose the space of states (2.1) in terms of representations of these algebras. In order to do so, we need to introduce some notation.

By $\bar{\mathfrak{g}}$ we mean the finite dimensional simple Lie algebra corresponding to $\mathfrak{g}_k$. We assume for convenience that $\bar{\mathfrak{g}}$ is simply laced; the generalisation to the non simply laced case is straightforward. The root lattice of $\bar{\mathfrak{g}}$ is denoted by Λ_R, and the weight lattice is $\Lambda_W \cong \Lambda_R^*$. A basis for the root lattice is given by the simple roots α_i, $i = 1, \ldots, \mathrm{rk}(\bar{\mathfrak{g}})$, all of whose length squares are equal to 2. The corresponding dual basis of the weight lattice is given by the fundamental weights Λ_i, i.e. $\Lambda_i \cdot \alpha_j = \delta_{ij}$. For the case of $\bar{\mathfrak{g}} = \mathfrak{su}(n+1)$ the inner products of the fundamental weights are

$$\Lambda_i \cdot \Lambda_j = \frac{i(n+1-j)}{n+1}\,, \tag{2.2}$$

where $i \leq j$. (For a general introduction to these matters see for example [16].)

The free bosons that make up $\mathfrak{h}$ give rise to an extended symmetry algebra,[1] whose representations are parametrised by $P_+^{\mathfrak{h}} = \Lambda_W / k\Lambda_R$; the corresponding representation spaces decompose as

$$\widehat{\mathcal{H}}_\mu^{\mathfrak{h}} = \bigoplus_{\delta \in k\Lambda_R} \mathcal{H}_{\mu+\delta}^{\mathfrak{h}}\,, \tag{2.3}$$

where $\mathcal{H}_{\mu+\delta}^{\mathfrak{h}}$ is the Fock space that is generated by the action of the $\mathrm{rk}(\bar{\mathfrak{g}})$ bosonic oscillators from a ground state $|\mu+\delta\rangle$ that is an eigenvector of the oscillator zero modes.

For the case of $\bar{\mathfrak{g}} = \mathfrak{su}(n+1)$ the modular S-matrix of this extended symmetry algebra is then

$$S_{\mu\mu'}^{\mathfrak{h}} = \frac{1}{\sqrt{n+1}\,k^{n/2}}\, e^{2\pi \mathrm{i} \frac{\mu \cdot \mu'}{k}}\,, \tag{2.4}$$

and it leads to the usual fusion rules

$$N^{\mathfrak{h}}{}_{\mu_1\mu_2}{}^{\mu_3} = \delta^{(k\Lambda_R)}_{\mu_3, \mu_1+\mu_2}\,. \tag{2.5}$$

The representations of the coset algebra $\mathfrak{g}_k/\mathfrak{h}$ are labelled by pairs (λ, μ), where $\lambda \in P_+^k$, $\mu \in P_+^{\mathfrak{h}}$, and $\lambda - \mu \in \Lambda_R$. Not all of these pairs of representations are inequivalent. For example, for $\mathfrak{g}_k = \widehat{\mathfrak{su}}(n+1)_k$ two pairs (λ_1, μ_1) and (λ_2, μ_2) define the same coset representation if and only if

$$\lambda_1 = J^l \lambda_2 \qquad \text{and} \qquad \mu_1 = \mu_2 + klJ' \pmod{k\Lambda_R}\,, \tag{2.6}$$

where J is the simple current acting on affine weights $(\lambda_0; \lambda_1, \ldots, \lambda_n)$ of $\widehat{\mathfrak{su}}(n+1)_k$ by $J(\lambda_0; \lambda_1, \ldots, \lambda_n) = (\lambda_n; \lambda_0, \ldots, \lambda_{n-1})$; the corresponding simple current of the free bosonic theory acts on the weights by addition of the nth fundamental weight $J' = \Lambda_n$. We denote the group of these field identifications by G_{id}; by construction it has order $n+1$.

Let us denote by $[\lambda, \mu]$ the equivalence class of such pairs, i.e. the orbit $[\lambda, \mu] = G_{id}(\lambda, \mu)$. Then the S-matrix of the coset theory is simply

$$S^{\mathfrak{g}/\mathfrak{h}}_{[\lambda,\mu]\,[\lambda',\mu']} = (n+1)\, S^{\mathfrak{g}}_{\lambda\lambda'}\, \bar{S}^{\mathfrak{h}}_{\mu\mu'}\,. \tag{2.7}$$

[1] The symmetry algebra is extended by vertex operators associated to elements in $k\Lambda_R$. The latter have integer conformal weight since the conformal weight of the state associated to $\mu \in \Lambda_W$ in the free boson theory is $\mu^2/2k$.

It is easy to check that this matrix is well defined on the equivalence classes, and that it is unitary; in particular, this implies that the above field identifications are in fact the only field identifications.

For each $\lambda \in P_+^k$ we can decompose the corresponding space of states as

$$\mathcal{H}_\lambda = \bigoplus_{\mu \in \Lambda_W} \mathcal{H}_{[\lambda,\mu+k\Lambda_R]} \otimes \mathcal{H}^{\mathfrak{h}}_\mu \,, \tag{2.8}$$

where the sum runs over all μ for which $\lambda - \mu \in \Lambda_R$, and $\mathcal{H}_{[\lambda,\mu+k\Lambda_R]}$ is the representation of the coset algebra. The total space of states therefore has the decomposition

$$\mathcal{H} = \bigoplus_{\lambda \in P_+^k} \bigoplus_{\mu,\bar{\mu} \in \Lambda_W} \left(\mathcal{H}_{[\lambda,\mu+k\Lambda_R]} \otimes \bar{\mathcal{H}}_{[\lambda^*,\bar{\mu}+k\Lambda_R]}\right) \otimes \left(\mathcal{H}^{\mathfrak{h}}_\mu \otimes \bar{\mathcal{H}}^{\mathfrak{h}}_{\bar{\mu}}\right), \tag{2.9}$$

where again $\lambda - \mu$, $\lambda^* - \bar{\mu} \in \Lambda_R$.

3. Boundary conditions

As we have mentioned before, we are interested in constructing boundary states of the above WZW model that only preserve in general the subalgebra $\mathfrak{h}$, as well as the coset algebra $\mathfrak{g}_k/\mathfrak{h}$. Obviously the usual 'untwisted' and 'twisted' boundary states (whose construction is well understood) are special examples of such boundary states. We want to generalise their construction by changing the gluing conditions for the free bosons that make up the Cartan subalgebra $\bar{\mathfrak{h}}$; in particular, we want to consider different combinations of Neumann and Dirichlet boundary conditions for these bosons.

More specifically, the usual D-branes of the WZW model are characterised by the gluing conditions

$$\left(J^a_n + \omega(\bar{J}^a_{-n})\right) ||\omega\rangle\rangle = 0\,, \tag{3.1}$$

where $\omega = \mathrm{id}$ for the untwisted branes, and ω is the non-trivial outer automorphism for the case of the twisted branes. (For A_n the outer automorphism is simply charge conjugation, $\omega = C$.)

In terms of the subalgebra $\mathfrak{g}_k/\mathfrak{h}$, these gluing conditions are

$$\left(S_n - (-1)^{h_S}\omega(\bar{S}_{-n})\right) ||\omega\rangle\rangle = 0\,, \tag{3.2}$$

where S_n denote the modes of a field in the coset algebra $\mathfrak{g}_k/\mathfrak{h}$, and ω is the induced automorphism on this algebra. In particular, there are therefore (at least) two different gluing conditions that can be imposed on the coset algebra.

The gluing conditions for the free bosons that make up $\mathfrak{h}$ are

$$\left(H^i_n + \sigma^{ij}\bar{H}^j_{-n}\right) ||\sigma,\omega\rangle\rangle = 0\,, \tag{3.3}$$

where σ is an orthogonal matrix; for $\omega = \mathrm{id}$, $\sigma = \mathbf{1}$, while for $\omega = C$, $\sigma = -\mathbf{1}$. We want to generalise this construction by considering more general σ for each given choice of ω, i.e. we want to construct the boundary states that are characterised by (3.2) and (3.3), but that do not necessarily satisfy (3.1).

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For the following it will be convenient to choose a suitable orthogonal basis for the Cartan subalgebra $\bar{\mathfrak{h}}$, and to restrict the construction to those σ that are diagonal with respect to this basis. As before, let Λ_i, $i = 1, \ldots, n$ be the fundamental weights of $\mathfrak{su}(n+1)$. We define

$$\tilde{\Lambda}_n = \Lambda_n \tag{3.4}$$

$$\tilde{\Lambda}_j = \Lambda_j - \frac{j}{j+1}\Lambda_{j+1}\,, \qquad 1 \leq j < n\,. \tag{3.5}$$

One then easily checks that these weights are pairwise orthogonal,

$$\tilde{\Lambda}_i \cdot \tilde{\Lambda}_j = \delta_{ij}\frac{j}{j+1}\,. \tag{3.6}$$

If one considers the filtration of algebras $\mathfrak{su}(2) \subset \mathfrak{su}(3) \cdots \subset \mathfrak{su}(n) \subset \mathfrak{su}(n+1)$, where $\mathfrak{su}(l+1)$ is generated by the first l fundamental roots, then $\tilde{\Lambda}_j$ is just the jth fundamental weight of $\mathfrak{su}(j+1)$. This is a consequence of the fact that

$$\alpha_j \cdot \tilde{\Lambda}_l = \begin{cases} 0 & \text{if } j < l \text{ or } j > l+1 \\ 1 & \text{if } j = l \\ -\dfrac{l}{l+1} & \text{if } j = l+1\,. \end{cases} \tag{3.7}$$

We shall consider σ that are diagonal in this basis. Since σ is orthogonal, the possible eigenvalues of σ are just ± 1. We choose the convention that $\sigma\tilde{\Lambda}_j = s_j\tilde{\Lambda}_j$ where $s_j = \pm 1$. The action of σ on a general $\bar{\mu} \in \Lambda_W$ is then

$$\sigma\,\bar{\mu} = \sum_j s_j \frac{j+1}{j}(\tilde{\Lambda}_j \cdot \bar{\mu})\tilde{\Lambda}_j\,, \tag{3.8}$$

where we have used that $\bar{\mu}$ can be written as

$$\bar{\mu} = \sum_j \frac{j+1}{j}(\tilde{\Lambda}_j \cdot \bar{\mu})\,\tilde{\Lambda}_j\,. \tag{3.9}$$

The first step of the construction consists of identifying the Ishibashi states that satisfy (3.2) and (3.3). The analysis will depend on whether ω is the trivial automorphism or charge conjugation, and we will therefore have to consider these two cases in turn.

3.1 The untwisted construction

Let us first consider the case where $\omega = \mathrm{id}$. It then follows from (2.9) that we get an Ishibashi state for every $(\lambda, \mu, \bar{\mu})$ for which $\mu - \lambda, \bar{\mu} - \lambda^* \in \Lambda_R$ and

$$\mu = -\sigma\bar{\mu}\,, \qquad \mu = -\bar{\mu} \pmod{k\Lambda_R}\,. \tag{3.10}$$

By combining these two conditions we therefore get precisely one Ishibashi state for each $\bar{\mu} \in \Lambda_W$ for which

$$(\mathbf{1} - \sigma)\,\bar{\mu} \in k\Lambda_R\,. \tag{3.11}$$

If $\bar\mu$ satisfies (3.11) then $\mu := -\sigma\bar\mu \in \Lambda_W$. Furthermore, since $\mu+\bar\mu \in \Lambda_R$, $\lambda-\mu\in\Lambda_R$. Let us denote the corresponding Ishibashi state by $|\lambda,\bar\mu\rangle\rangle$. Our ansatz for the boundary state is then

$$||\sigma,\nu\rangle\rangle = \sqrt{|\Gamma_\sigma|}\sum_{\lambda\in P_+^k}\frac{S_{\lambda\nu}}{\sqrt{S_{\lambda 0}}}\sum_{\bar\mu+\lambda\in\Lambda_R}{}' \, e^{\mathrm{i}\,\theta\cdot\bar\mu}\,|\lambda,\bar\mu\rangle\rangle\,, \tag{3.12}$$

where the last sum is restricted to the solutions of (3.11) and $|\Gamma_\sigma|$ is the order of a finite abelian group Γ_σ that will be defined below. The parameter θ describes the position or Wilson line, and we shall set $\theta=0$ from now on. The S-matrix here is the S-matrix of $\mathfrak{g}_k$, and we have used that $\bar\mu-\lambda^*\in\Lambda_R$ if and only if $\bar\mu+\lambda\in\Lambda_R$.

In order to calculate the overlap between these boundary states, it is important to understand how to characterise the solutions of (3.11). Using the explicit formulae (3.8) and (3.9) we can rewrite (3.11) as

$$\frac{1}{k}\sum_j(1-s_j)\frac{j+1}{j}\,(\tilde\Lambda_j\cdot\bar\mu)\,\tilde\Lambda_j\in\Lambda_R\,. \tag{3.13}$$

A vector $\bar\gamma$ is in the root lattice if both $\Lambda_n\cdot\bar\gamma\in\mathbb{Z}$, and if $\bar\gamma$ is in the weight lattice, i.e. if $\alpha_j\cdot\bar\gamma\in\mathbb{Z}$ for all j. Dotting the above equation with Λ_n we therefore obtain the constraint that

$$(1-s_n)\,\Lambda_n\cdot\bar\mu\in k\,\mathbb{Z}\,. \tag{3.14}$$

If we define the n-ality of a weight $\mu\in\Lambda_W$ by $t(\mu)=\sum_j j\mu_j$, then (3.14) can simply be written as

$$(1-s_n)\,t(\bar\mu)\in k\,(n+1)\,\mathbb{Z}\,. \tag{3.15}$$

If $s_n=+1$, this condition is empty, but if $s_n=-1$ it implies that there are $\lambda\in P_+^k$ for which no Ishibashi state $|\lambda,\bar\mu\rangle\rangle$ exists. This suggests that the construction will break down for $s_n=-1$, and this is indeed what will become apparent below. (If $s_n=-1$ one has to use the twisted construction instead that will be described in the following subsection.) For now we therefore assume that $s_n=+1$.

This leaves us with analysing the condition that (3.13) is actually a weight. Dotting the equation by α_j and using (3.7) we obtain

$$\frac{1}{k}\left[(1-s_j)\frac{j+1}{j}\,(\tilde\Lambda_j\cdot\bar\mu)-(1-s_{j-1})\,(\tilde\Lambda_{j-1}\cdot\bar\mu)\right]\in\mathbb{Z}\,. \tag{3.16}$$

Since $\alpha_j=2\Lambda_j-\Lambda_{j-1}-\Lambda_{j+1}$ we can rewrite this condition as

$$\left[(1-s_j)\,\alpha_j+(s_{j-1}-s_j)\,\tilde\Lambda_{j-1}\right]\cdot\bar\mu\in k\,\mathbb{Z}\,. \tag{3.17}$$

This condition has to hold for all $j=1,\ldots,n$. As is explained in the appendix, one can construct a projector P_σ that projects any state in $\mathcal{H}$ onto the components for which $\bar\mu$ satisfies (3.11). This projector can be written as

$$P_\sigma=\frac{1}{|\Gamma_\sigma|}\sum_{v\in\Gamma_\sigma}\exp\left[2\pi\mathrm{i}\,v\cdot\bar H_0\right]\,, \tag{3.18}$$

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where Γ_σ is a finite abelian group that is a quotient of a lattice of shift vectors by roots. It is important here that Γ_σ does not intersect the weight lattice Λ_W; all this is explained in detail in the appendix.

It is now easy to calculate the overlaps between two branes of the form (3.12). We find

$$\begin{aligned}\langle\langle \sigma,\nu_1 || q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}} || \sigma,\nu_2\rangle\rangle &= |\Gamma_\sigma| \sum_{\lambda\in P_+^k} \frac{S^*_{\lambda\nu_1} S_{\lambda\nu_2}}{S_{\lambda 0}} \sum_{\mu-\lambda\in\Lambda_R}{}' \chi_{[\lambda,\mu]}(\tau)\, \chi^{\mathfrak{h}}_\mu(\tau) \\ &= \sum_{\lambda\in P_+^k} \frac{S^*_{\lambda\nu_1} S_{\lambda\nu_2}}{S_{\lambda 0}} \sum_{v\in\Gamma_\sigma} \chi_\lambda(\tau,v,0)\,, \end{aligned} \tag{3.19}$$

where $\chi_\lambda(\tau,v,t)$ is the unspecialised affine character,

$$\chi_\lambda(\tau,v,t) = e^{-2\pi\mathrm{i}\,tk}\ \mathrm{Tr}_{\mathcal{H}_\lambda}\left(e^{2\pi\mathrm{i}\,\tau(L_0-\frac{c}{24})}\, e^{2\pi\mathrm{i}\,v\cdot H_0}\right). \tag{3.20}$$

Under a modular transformation, the unspecialised characters $\chi_\lambda(\tau,v,0)$ transform into the characters $\chi_{\lambda'+v}$ of representations $\lambda'+v$, which are *twisted* by inner twists associated to v [17, 18]:

$$\begin{aligned}\chi_\lambda\left(-\frac{1}{\tau},v,0\right) &= \sum_{\lambda'\in P_+^k} S_{\lambda\lambda'}\,\chi_{\lambda'}\left(\tau,\tau\, v,-\frac{\tau k v^2}{2}\right) \\ &= \sum_{\lambda'\in P_+^k} S_{\lambda\lambda'}\,\chi_{\lambda'+v}(\tau,0,0)\,. \end{aligned} \tag{3.21}$$

(For a clear introduction to twisted representations see for example [19].) The above cylinder diagram therefore becomes in the open string channel

$$\begin{aligned}\langle\langle \sigma,\nu_1 ||\, q^{\frac{1}{2}(L_0+\bar{L}_0)-\frac{c}{12}} || \sigma,\nu_2\rangle\rangle &= \sum_{\lambda\in P_+^k} \frac{S^*_{\lambda\nu_1} S_{\lambda\nu_2}}{S_{\lambda 0}} \sum_{v\in\Gamma_\sigma}\sum_{\lambda'\in P_+^k} S_{\lambda\lambda'}\,\chi_{\lambda'+v}(\tau,0,0) \\ &= \sum_{\lambda'\in P_+^k} N_{\nu_2\lambda'}{}^{\nu_1} \sum_{v\in\Gamma_\sigma} \chi_{\lambda'+v}(\tau,0,0)\,, \end{aligned} \tag{3.22}$$

where $N_{\nu_2\lambda'}{}^{\nu_1}$ are the fusion rules of $\mathfrak{g}_k$. In particular, this implies that these boundary states satisfy Cardy's condition. Since Γ_σ does not intersect the weight lattice, $\chi_{\mu+v}(\tau,0,0) = \chi_{\mu'+v'}(\tau,0,0)$ if and only if $\mu=\mu'$ and $v=v'$. Thus the different representations that appear on the right hand side of (3.22) are in fact all different.

These boundary states therefore still define a NIM-rep (for an introduction into these matters see for example [20]), the only difference being that now the non-negative matrices are also associated to twisted representations of $\mathfrak{g}_k$. Indeed, the above formula simply implies that for any λ that differs by a twist in Γ_σ from $\lambda'\in P_+^k$

$$\mathcal{N}_{\nu_2\lambda}{}^{\nu_1} = N_{\nu_2[\lambda]}{}^{\nu_1}\,, \tag{3.23}$$

where $[\lambda]$ is the unique untwisted representation that can be obtained from λ by a twist $v\in\Gamma_\sigma$. The fusion of two twisted representations λ and ν both of whose twists are inner, is simply the fusion product of $[\lambda]$ and $[\mu]$, twisted by the sum of the two twists [21]. Thus it is manifest that (3.23) still defines a NIM-rep. (In fact, this NIM-rep is simply the tensor product of the original NIM-rep of $\mathfrak{g}$, and the NIM-rep associated to Γ_σ.) Since all twists in question are inner, the dimension that should be associated to a twisted representation λ is simply the dimension of the corresponding untwisted representation $[\lambda]$. One can thus use the same arguments as in [8] to conclude that the charges that are associated to these D-branes must satisfy

$$\dim([\lambda])\, q_\nu = \sum_{\nu'} \mathcal{N}_{\nu\lambda}{}^{\nu'}\, q_{\nu'}\,. \tag{3.24}$$

Since the above NIM-rep agrees with the fusion rule, the same analysis as in [8] applies, and we conclude that each such family of D-branes contributes one summand $\mathbb{Z}_{d(G,k)}$ to the charge group.

3.2 The twisted construction

The analysis in the twisted case, i.e. when we choose $\omega = C$ in (3.2), is very similar. In this case we get an Ishibashi state for every $(\lambda, \mu, \bar{\mu})$ provided that

$$\lambda = \lambda^*\,, \qquad \mu = -\sigma\,\bar{\mu}\,, \qquad \mu = +\bar{\mu} \pmod{k\Lambda_R}\,. \tag{3.25}$$

If we define $\hat{\sigma} = -\sigma$, the last two conditions become

$$(\mathbf{1} - \hat{\sigma})\,\bar{\mu} \in k\Lambda_R\,, \tag{3.26}$$

and therefore agree formally with (3.11). However, $\hat{\sigma}$ differs by a sign from σ, and thus also the last sign of $\hat{\sigma}$, $\hat{s}_n$, is opposite to the last sign of σ, s_n. In particular, the choice $s_n = -1$ now leads to $\hat{s}_n = +1$; in this case, the analysis of the solution space to (3.26) is then identical to that of the previous section. If we tried to perform this construction for $s_n = +1$ we would run into the same difficulties as in the previous case with $s_n = -1$. In particular, as is explained in the appendix, it is then in general not possible to choose Γ_σ such that it does not intersect the weight lattice. In the following we therefore assume that $s_n = -1$ so that $\hat{s}_n = +1$.

In addition to the condition on $\bar{\mu}$ we now only get Ishibashi states that come from a self-conjugate λ. Thus the natural ansatz for our boundary states is now

$$||\sigma, x\rangle\rangle = \sqrt{|\Gamma_{\hat{\sigma}}|} \sum_{\lambda\in\mathcal{E}_\omega} \frac{\hat{S}_{\lambda x}}{\sqrt{S_{\lambda 0}}} \sum_{\bar{\mu}+\lambda\in\Lambda_R}{}' \, e^{\mathrm{i}\,\theta\cdot\bar{\mu}} |\lambda, \bar{\mu}\rangle\rangle\,, \tag{3.27}$$

where the last sum is restricted to the solutions of (3.26) and $|\Gamma_{\hat{\sigma}}|$ is the order of the finite abelian group $\Gamma_{\hat{\sigma}}$ that was defined before. Here $\mathcal{E}_\omega$ is the set of charge-conjugation invariant weights in P^k_+, and $\hat{S}_{\lambda x}$ is the twisted S-matrix that appears in the construction of the twisted D-branes (see for example [20]). These D-branes are now labelled by the twisted representations x of $\mathfrak{g}_k$. Using the same calculation as in the previous section we then find that these boundary states define indeed a NIM-rep, and that it is simply given by

$$\mathcal{N}_{x_2\lambda}{}^{x_1} = \mathcal{N}^C_{x_2[\lambda]}{}^{x_1}\,, \tag{3.28}$$

where $[\lambda]$ is the unique untwisted representation that can be obtained from λ by a twist $v \in \Gamma_{\hat{\sigma}}$, and the matrix $\mathcal{N}^C_{x_2\lambda}{}^{x_1}$ is the usual twisted NIM-rep of $\mathfrak{g}_k$ (see for example [20]). Using the results of [11] it then follows again that each such family of D-branes contributes one summand $\mathbb{Z}_{d(G,k)}$ to the charge group.

4. Concluding remarks

In the previous section we have constructed one set of D-branes for each orthogonal real matrix σ that is diagonal in the basis defined by $\tilde{\Lambda}_j$. Depending on the value $s_n = \pm 1$, the construction resulted in the untwisted or twisted NIM-rep. Since the twisted NIM-rep gives rise to the same charges as the untwisted NIM-rep [11], each of these constructions leads to a summand $\mathbb{Z}_{d(G,k)}$ in the charge group. There are $\mathrm{rk}(\bar{\mathfrak{g}})$ different signs in the definition for σ, and thus we get in total $2^{\mathrm{rk}(\bar{\mathfrak{g}})}$ constructions.

Not all of these choices are inequivalent though. There is precisely one non-trivial element of the Weyl group of $\bar{\mathfrak{g}}$ for which all $\tilde{\Lambda}_i$ are eigenvectors: it is the Weyl-reflection corresponding to α_1 which acts as -1 on $\tilde{\Lambda}_1$, while leaving all other $\tilde{\Lambda}_i$ invariant. This Weyl transformation therefore maps a brane to one with opposite s_1; strictly speaking, it also modifies the gluing conditions on the coset algebra, but it seems very plausible that two branes that have the same gluing condition on $\mathfrak{h}$ will in fact carry the same charge. Thus there are in fact only $2^{\mathrm{rk}(\bar{\mathfrak{g}})-1}$ branes with different charges, each of which leads to a summand of $\mathbb{Z}_{d(G,k)}$. This accounts precisely for the charges that were determined using K-theory arguments.

It seems very difficult to prove rigorously that two given branes carry different charges, i.e. lie on different sheets of the moduli space. However, it seems plausible that branes that have different gluing conditions on $\mathfrak{h}$ (up to the Weyl symmetry mentioned before) are in fact inequivalent. [After all, in order to preserve supersymmetry the fermions have to satisfy the corresponding gluing conditions, and these boundary states therefore couple to different RR ground states.] It would nevertheless be very interesting to establish this from first principles (for example by finding a numerical invariant that characterises the different sheets).

It would also be interesting to understand the geometrical interpretation of these branes. We have found another realisation [22] of these D-brane charges along the lines of [23], for which the geometrical interpretation is more obvious. We believe that these D-branes describe another point of each sheet of the moduli space.

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A. The projector

In this appendix we supply some technical details, completing the argument of section 3.

Recall that in our construction we retain only those weights $\bar{\mu}$ obeying (3.17) for all $j = 1, \ldots, n$. Define

$$v_j = \frac{1}{k}\left((1-s_j)\alpha_j + (s_{j-1}-s_j)\tilde{\Lambda}_{j-1}\right). \tag{A.1}$$

Then for each j define the projector

$$P_j = \frac{1}{L_j}\sum_{\ell=1}^{L_j} \exp[2\pi \mathrm{i}\, \ell v_j \cdot \bar{H}_0]\,, \tag{A.2}$$

where

$$L_j = \begin{cases} kj & \text{if } s_j \neq s_{j-1} \\ k & \text{if } s_j = s_{j-1} = -1 \\ 1 & \text{if } s_j = s_{j-1} = +1\,, \end{cases} \tag{A.3}$$

and $\bar{H}_0$ is the operator $\bar{H}_0|\mu,\bar{\mu}\rangle = \bar{\mu}|\mu,\bar{\mu}\rangle$. Clearly, $P_j|\mu,\bar{\mu}\rangle = |\mu,\bar{\mu}\rangle$ or 0, depending on whether or not $\bar{\mu}$ satisfies the jth equation (3.16). Thus the desired projector (3.18) is $P_\sigma = \prod_j P_j$.

We can describe P_σ additively as follows. Consider the quotient $\Gamma_\sigma = \Lambda_v/(\Lambda_v \cap \Lambda_R)$ of the $\mathbb{Z}$-span Λ_v of the vectors v_j, by its intersection with the root lattice Λ_R. Then Γ_σ is a finite abelian group, and P_σ can be written as in (3.18).

Establishing the NIM-rep property in section 3 required that we verify that P_σ produces only twisted representations. More precisely, we need to verify that, for any $v \in \Lambda_v$, $v \in \Lambda_W$ (the weight lattice) implies $v \in \Lambda_R$. This is easy to see. Recall that it suffices to consider $s_n = +1$. Suppose $v \in \Lambda_v \cap \Lambda_W$. Then $v \in \Lambda_R$ if and only if $\Lambda_n \cdot v \in \mathbb{Z}$. We compute

$$\sum_j \ell_j v_j \cdot \Lambda_n = \frac{\ell_n\,(1-s_n)}{k} = 0\,, \tag{A.4}$$

and the claim follows.

The order of the actual group Γ_σ is a factor of the product of the L_j. In order to compute the actual order of the group Γ_σ, we observe that a linear combination $\sum_i \ell_i v_i$ is trivial in Γ_σ if and only if it is an element of $k\Lambda_R$. In particular, it must therefore satisfy

$$\Lambda_j \cdot \left(\sum_i \ell_i v_i\right) \in k\mathbb{Z} \tag{A.5}$$

for each j. It is easy to see that this leads to a triangular set of conditions in the ℓ_i. In particular, for large k the order of Γ_σ grows like

$$|\Gamma_\sigma| \sim k^{n-m}\,, \tag{A.6}$$

where m is the number of j for which $s_{j-1} = s_j = +1$, $j = 1, \ldots, n$ with $s_0 = +1$.

Incidentally, this is the place where we see explicitly that the construction of section 3 collapses when $s_n = -1$ (or equivalently if we do not choose the gluing condition (3.2) for

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the coset algebra depending on the value of s_n). When $s_n = -1$, the requirement $v \in \Lambda_R$ demands that $2\ell_n \in k\mathbb{Z}$. This will not be satisfied in general. For a concrete example take n even and $\sigma = \mathrm{diag}(+1, +1, \ldots, +1, -1)$; $v = \ell_n v_n$ will lie in $\Lambda_v \cap \Lambda_W$ when k divides $2\ell_n(n+1)/n$, which certainly does not force k to divide $2\ell_n$. Thus, when $s_n = -1$, the construction will typically fail to produce a NIM-rep.

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